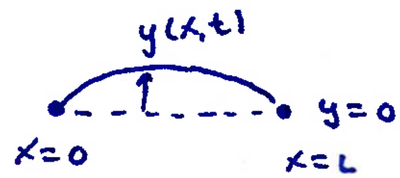
Wave Eq. (part 2)

$y_{tt} = a^2 y_{xx}$ $\quad 0 < x < L \quad t > 0$



$y(0,t) = y(L,t) = 0$ ends fixed

$y(x,0) = f(x)$ initial displacement

$y_t(x,0) = g(x)$ initial velocity

last time: $g(x) = 0$ (no initial velocity)

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

today: $f(x) = 0$ (no initial displacement)

$g(x) \neq 0$ some initial velocity ("Problem B") same basic idea: $y(x,t) = X(x)T(t)$

$\vdots$

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

two ODEs: $X'' + \lambda X = 0$

$T'' + a^2\lambda T = 0$

same BCs: $y(0,t) = 0 \to X(0) = 0$

$y(L,t) = 0 \to X(L) = 0$

same spatial solution: $\lambda_n = \dfrac{n^2\pi^2}{L^2}$ $n = 1, 2, 3, \ldots$

$X_n = \sin\left(\dfrac{n\pi x}{L}\right)$

$$T'' + a^2\lambda T = 0$$

$$T'' + \frac{a^2 n^2\pi^2}{L^2} T = 0$$

$$T(t) = A\cos\left(\frac{n\pi a}{L}t\right) + B\sin\left(\frac{n\pi a}{L}t\right)$$

IC: $y(x,0) = 0 \rightarrow X(x)T(0) = 0 \rightarrow T(0) = 0$ (last time: $T'(0)=0$)

here, $A = 0$

so, $T_n = \sin\left(\frac{n\pi a}{L}t\right)$ (Problem A has $\cos\left(\frac{n\pi a}{L}t\right)$)

for each n, $y_n = \sin\left(\frac{n\pi a}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)$

general solution: $\boxed{y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)}$

last IC: $y_t(x,0) = g(x)$ initial velocity

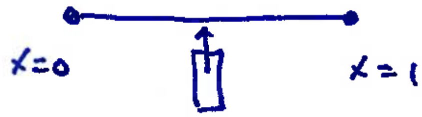
$$y_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi a}{L} B_n \cos\left(\frac{n\pi a}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)$$

$$g(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi a}{L}B_n\right)\sin\left(\frac{n\pi}{L}x\right)$$ sine series w/ coefficients $\frac{n\pi a}{L}B_n$

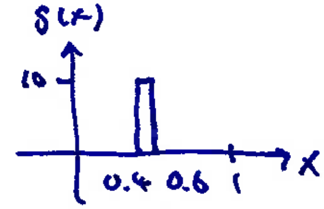
$$\frac{n\pi a}{L}B_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$\boxed{B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx}$$

example A string w/ $L=1$ and $a=1$ is initially at rest and is struck w/ a hammer of width 0.2 at the center w/ upward velocity of 10.



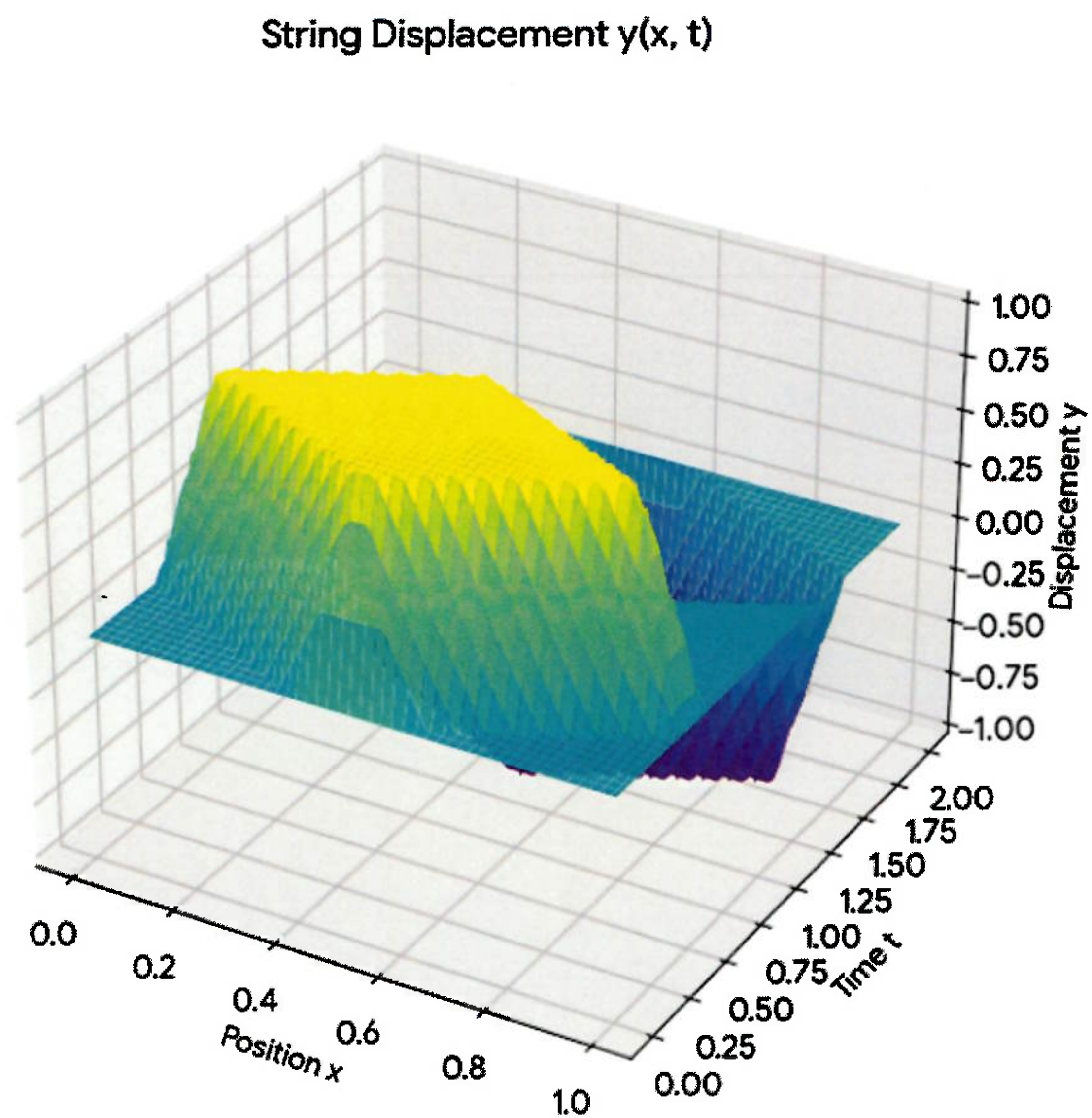
initial velocity $g(x) = \begin{cases} 10 & 0.4 < x < 0.6 \\ 0 & \text{else} \end{cases}$

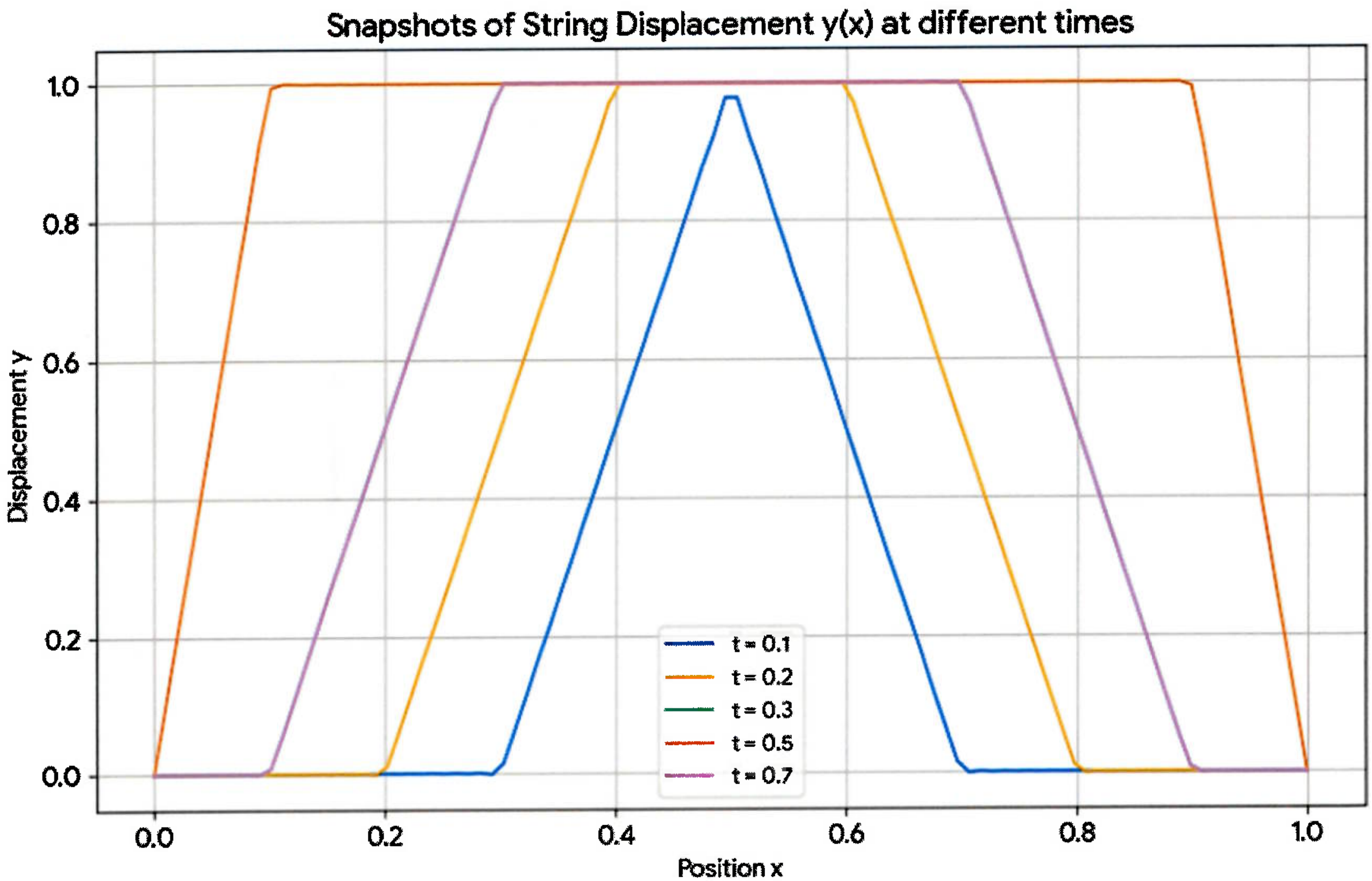


⋮

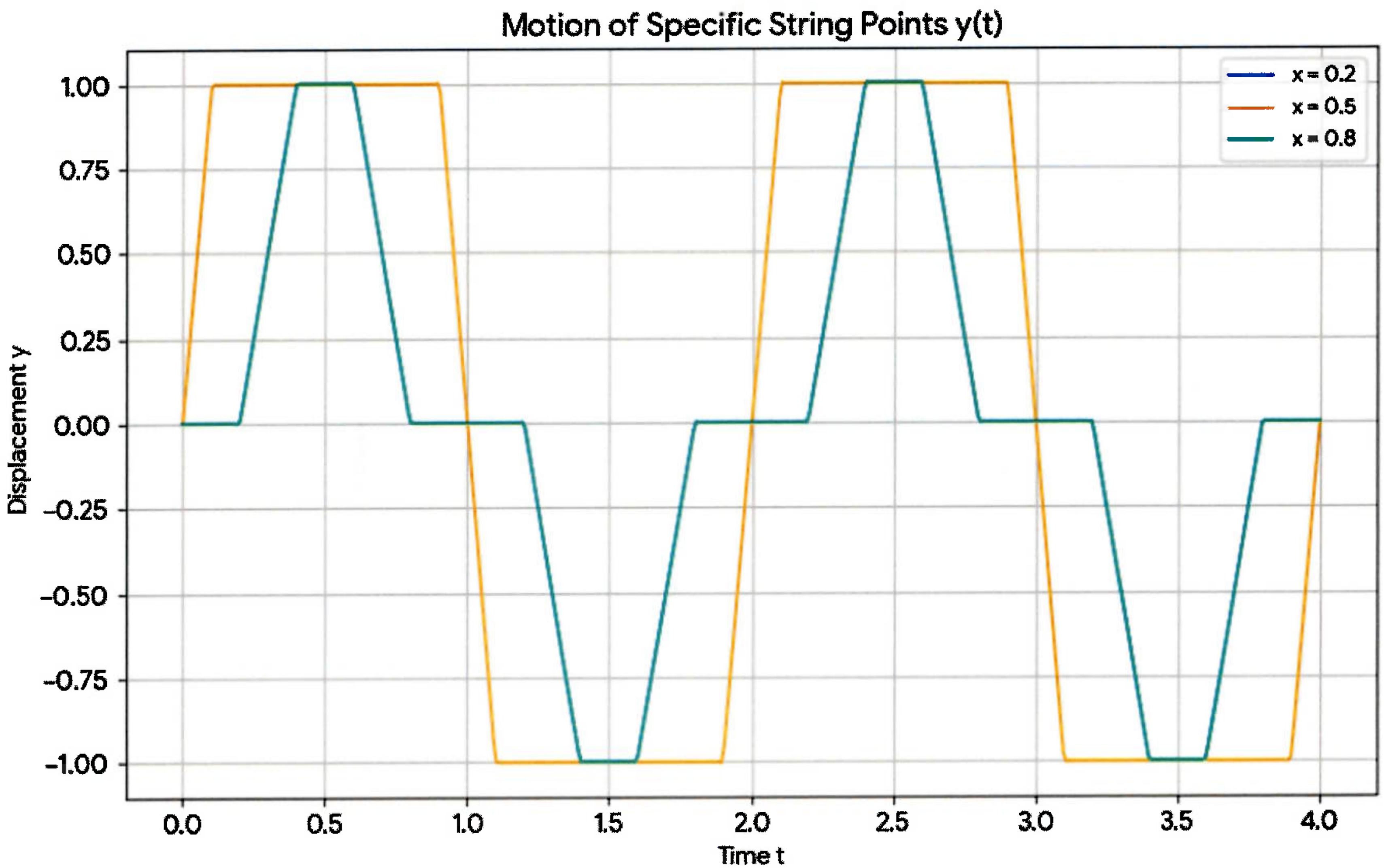
$$y(x,t) = \sum_{n=1}^{\infty} \frac{20}{n^2\pi^2}\left[\cos(0.4n\pi) - \cos(0.6n\pi)\right] \sin(\boxed{n\pi}\,t)\,\sin(n\pi x)$$

frequency of each mode (n)

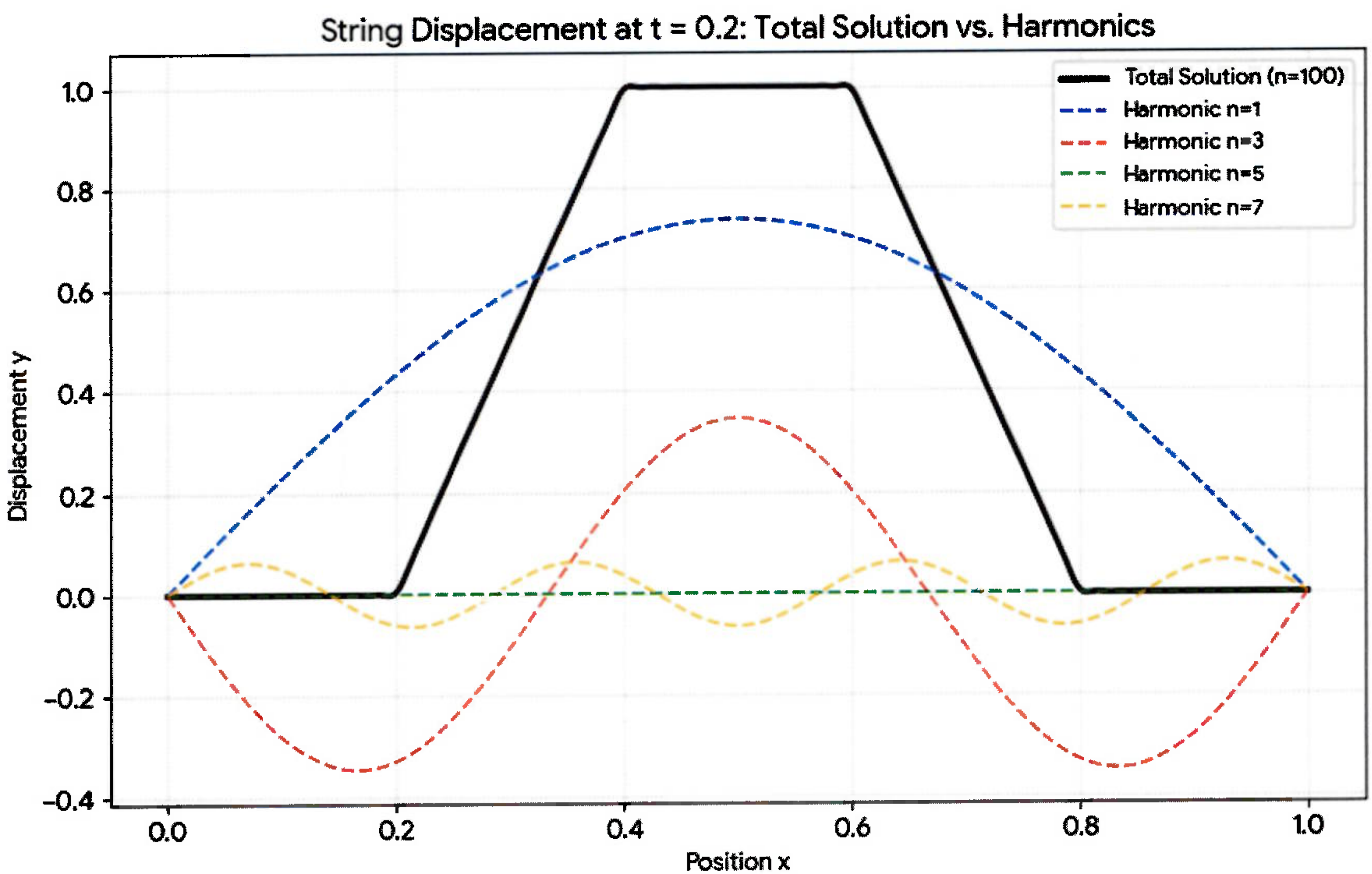

String Displacement y(x, t)
1.00
0.75
0.50
0.25
0.00
-0.25
-0.50
-0.75
-1.00
Displacement y
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
Time t
0.0
0.2
0.4
0.6
0.8
1.0
Position x



Snapshots of String Displacement y(x) at different times
Displacement y
Position x
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
t = 0.1
t = 0.2
t = 0.3
t = 0.5
t = 0.7



Motion of Specific String Points y(t)
x = 0.2
x = 0.5
x = 0.8
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
Displacement y
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
Time t



String Displacement at t = 0.2: Total Solution vs. Harmonics
Total Solution (n=100)
Harmonic n=1
Harmonic n=3
Harmonic n=5
Harmonic n=7
Displacement y
Position x
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0.0
0.2
0.4
0.6
0.8
1.0

there are no even harmonics in that solution (even n's)

why?

$n=1$

$n=2$ center at rest for all t

$n=3$

$n=4$ center at rest for all t

since we ~~be~~ are hitting the string at center, the center of string must move, but even harmonics have center at rest so all even harmonics are "destroyed" (same if plucked)

on real piano, the hammer usually strikes at $L/7$ or $L/9$

↓

eliminate multiples of 7 (dissonant)

a struck string starts at zero displacement → no initial sound
(small delay before sound is made)

Problem A : displacement only

Problem B : velocity only

$y_{tt} = a^2 y_{xx}$ is linear

so general problem : Problem A + Problem B (just add them)

for wind instrument, no string, sound is from pressure waves

e.g. $$\frac{\partial^2 p}{\partial t^2} = a^2 \frac{\partial^2 p}{\partial x^2}$$ same equation!